Settling Multiple Debts Efficiently: Problems

Tom Verhoeff
Faculty of Mathematics and Computing Science
Eindhoven University of Technology
Email: T.Verhoeff@TUE.NL

June 2004

Abstract

I present a series of problems related to [1] for use in the class room.

Introduction

A group of friends lend each other money throughout the year. They carefully record each transaction. When Alice lends 10 euro to Bob, this is recorded as Alice $\xrightarrow{10}$ Bob.

At the end of the year they wish to settle their debts. How should they transfer money so as to settle all debts? Of course, they could reverse the action of each recorded loan separately. However, often there is a better way.

The problems posed below concern efficient ways of settling all debts. Try to minimize the number of transfers and the total amount transferred. We consider only settling schemes where money is transferred between a pair of persons.

Problems

1. After the loans

$$\begin{array}{lcl} \text{Alice} & \xrightarrow{10} & \text{Bob} \\ \text{Carol} & \xrightarrow{10} & \text{Dick} \end{array}$$

is this a proper settlement:

$$\begin{array}{lcl} \text{Bob} & \xrightarrow{10} & \text{Carol} \\ \text{Dick} & \xrightarrow{10} & \text{Alice} \end{array}$$

(involving 2 transfers for a total amount of 20 euro)?

2. How to settle efficiently the loans:

$$\begin{array}{lcl} \text{Alice} & \xrightarrow{10} & \text{Bob} \\ \text{Alice} & \xrightarrow{10} & \text{Carol} \\ \text{Bob} & \xrightarrow{10} & \text{Carol} \end{array}$$

3. How to settle efficiently the loans:

$$\begin{array}{lcl} \text{Alice} & \xrightarrow{10} & \text{Bob} \\ \text{Alice} & \xrightarrow{20} & \text{Carol} \\ \text{Bob} & \xrightarrow{15} & \text{Carol} \\ \text{Carol} & \xrightarrow{25} & \text{Alice} \end{array}$$

4. How to settle efficiently the loans:

$$\begin{array}{lcl} \text{Alice} & \xrightarrow{10} & \text{Bob} \\ \text{Alice} & \xrightarrow{10} & \text{Carol} \\ \text{Bob} & \xrightarrow{10} & \text{Carol} \\ \text{Carol} & \xrightarrow{20} & \text{Alice} \end{array}$$

5. How to settle efficiently the loans:

$$\begin{array}{lcl} \text{Alice} & \xrightarrow{10} & \text{Bob} \\ \text{Alice} & \xrightarrow{20} & \text{Dick} \\ \text{Carol} & \xrightarrow{30} & \text{Bob} \\ \text{Carol} & \xrightarrow{40} & \text{Dick} \end{array}$$

Can you find another (optimal) way?

6. How to settle efficiently the loans:

$$\begin{array}{lcl} \text{Alice} & \xrightarrow{20} & \text{Bob} \\ \text{Bob} & \xrightarrow{10} & \text{Carol} \\ \text{Alice} & \xrightarrow{60} & \text{Dick} \\ \text{Dick} & \xrightarrow{30} & \text{Carol} \\ \text{Carol} & \xrightarrow{20} & \text{Alice} \end{array}$$

7. How can the settlement

$$\begin{array}{lcl} \text{Alice} & \xrightarrow{10} & \text{Carol} \\ \text{Carol} & \xrightarrow{60} & \text{Dick} \\ \text{Dick} & \xrightarrow{30} & \text{Bob} \end{array}$$

be improved to minimize the total amount transferred?

8. How can the settlement

$$\begin{array}{lcl} \text{Alice} & \xrightarrow{20} & \text{Bob} \\ \text{Alice} & \xrightarrow{30} & \text{Dick} \\ \text{Carol} & \xrightarrow{30} & \text{Bob} \end{array}$$

be improved to minimize the number of transfers?

9. Describe a general scheme for settling all debts among N persons in at most $N-1$ transfers and with a minimal total amount transferred. It is not necessary to minimize the number of transfers.

10. How can the six balances

$$\begin{array}{rr} -120 & +110 \\ -60 & +90 \\ -50 & +30 \end{array}$$

be settled optimally?

11. How can the twelve balances

$$\begin{array}{rr} -301 & +900 \\ -300 & +203 \\ -299 & +100 \\ -297 & +99 \\ -202 & +98 \\ -2 & +1 \end{array}$$

be settled optimally?

Solutions

1. Yes, although that may seem counterintuitive at first.

   Of importance is the *balance* of each person: the total amount lend to others minus the total amount borrowed from others.

   Alice and Carol have a credit of 10 euro (balance +10), whereas Bob and Dick have a debt of 10 euro (balance −10). If the friends don't mind what money (as concrete physical objects) they get from whom, but only care about the (abstract) amount, then the proposed settlement is equivalent to the more obvious settlement:

   $$\begin{array}{lcl} \text{Bob} & \xrightarrow{10} & \text{Alice} \\ \text{Dick} & \xrightarrow{10} & \text{Carol} \end{array}$$

   It has the same number of transfers and the same total amount transferred.

2. The balances are:

Alice	+20
Bob	0
Carol	−20

   Hence an 'optimal' settlement is

   $$\text{Carol} \xrightarrow{20} \text{Alice}$$

   with a single transfer of 20 euro.

3. The balances are:

Alice	+5
Bob	+5
Carol	−10

   Hence an 'optimal' settlement is

   $$\begin{array}{lcl} \text{Carol} & \xrightarrow{5} & \text{Alice} \\ \text{Carol} & \xrightarrow{5} & \text{Bob} \end{array}$$

4. The balances are:

Alice	0
Bob	0
Carol	0

   Hence, no transfers are required at all.

5. The balances are:

Alice	+30
Bob	−40
Carol	+70
Dick	−60

Hence an 'optimal' settlement is

$$\begin{array}{lcl} \text{Bob} & \xrightarrow{30} & \text{Alice} \\ \text{Bob} & \xrightarrow{10} & \text{Carol} \\ \text{Dick} & \xrightarrow{60} & \text{Carol} \end{array}$$

Also 'optimal' is

$$\begin{array}{lcl} \text{Bob} & \xrightarrow{40} & \text{Carol} \\ \text{Dick} & \xrightarrow{30} & \text{Alice} \\ \text{Dick} & \xrightarrow{30} & \text{Carol} \end{array}$$

Both involve three transfers for a total amount of 100 euro. These are the only two optimal settlements.

6. The balances are:

Alice	+60
Bob	−10
Carol	−20
Dick	−30

Hence an 'optimal' settlement is

$$\begin{array}{lcl} \text{Bob} & \xrightarrow{10} & \text{Alice} \\ \text{Carol} & \xrightarrow{20} & \text{Alice} \\ \text{Dick} & \xrightarrow{30} & \text{Alice} \end{array}$$

with three transfers for a total amount of 60 euro.

7. The balances before the settlement were:

Alice	−10
Bob	+30
Carol	−50
Dick	+30

Hence an 'optimal' settlement is for example

$$\begin{array}{lcl} \text{Alice} & \xrightarrow{10} & \text{Bob} \\ \text{Carol} & \xrightarrow{20} & \text{Bob} \\ \text{Carol} & \xrightarrow{30} & \text{Dick} \end{array}$$

with three transfers for a total amount of only 60 euro (instead of 100). Bob and Dick can also be exchanged.

Note that the settlement

$$\begin{array}{lcl} \text{Alice} & \xrightarrow{10} & \text{Carol} \\ \text{Carol} & \xrightarrow{30} & \text{Bob} \\ \text{Carol} & \xrightarrow{30} & \text{Dick} \end{array}$$

is an improvement with a total transferred amount of 70 euro, but it is not ‘optimal’.

8. The balances before the settlement were:

Alice	-50
Bob	$+50$
Carol	-30
Dick	$+30$

Hence an ‘optimal’ settlement is

$$\begin{array}{lcl} \text{Alice} & \xrightarrow{50} & \text{Bob} \\ \text{Carol} & \xrightarrow{30} & \text{Dick} \end{array}$$

with only two transfers (instead of three) for a total amount of 80 euro.

9. **Step 1.** Determine the balance for each person.

   Observe that the sum of all balances equals zero.
   Let P be the total amount of positive balances, and N the total amount of negative balances. Hence, $P = -N$.
   The minimum total amount to be transferred equals P.

   Step 2. While there is still someone with a nonzero balance, do:

   - **Step 2a.** Select a person A with a negative balance $S < 0$, and a person B with a positive balance $T > 0$ (these exist).
   - **Step 2b.** Let M be the minimum of $-S$ and T. Hence, $M > 0$.
   - **Step 2c.** Include the transfer $A \xrightarrow{M} B$ in the settlement.
   - **Step 2d.** Increase the balance of A by M and decrease the balance of B by M (the total balance remains zero).

     Observe that after Step 2d, at least one of A and B now has balance zero.

   Step 3. All balances are zero, hence the included transfers settle all debts.

   The total amount transferred equals P, and hence is minimal. The repetition of Step 2 terminates, because in each iteration at least one nonzero balance is reduced to zero. Therefore, the number of transfers is at most N. In fact, it is at most $N - 1$, because the final two nonzero balances cancel each other in a single transfer.

   The settlement obtained by this algorithm is not guaranteed to have a minimal number of transfers. For example, the nonoptimal settlement of Problem 6 can be obtained from the balances via the algorithm above.

   It is not known how to minimize the number of transfers efficiently. This is a NP-hard problem, for which only algorithms are known that “try” an exponential number of possibilities.

10. First transferring between the closest pair -120 and $+110$ is not optimal. Instead, it is better to combine

$$\begin{array}{ll} -120 & \text{with } +90,\ +30 \\ -60,\ -50 & \text{with } +110 \end{array}$$

Within a zero-sum group of k persons, the debts can be settled in no more than $k-1$ transfers (using the algorithm above). Thus, if the initial group of N persons, can be split into g zero-sum groups, then at most $N-g$ transfers are needed for settling (verify this!). Each additional zero-sum group that can be created reduces the number of transfers by one.

Lesson: Minimizing the number of transfers is equivalent to maximizing the number of zero-sum groups, that is, the number of groups that can settle among themselves.

11. Canceling −301 with +203 and +98 yields a 10-transfer settlement, because the remaining nine balances cannot be split further into groups.

    Split into the following three groups of four balances is better:

−301, −300, −299	with +900
−202, −2	with +203, +1
−297	with +100, +99, +98

    Since each group can be settled in three transfers, this yields a 9-transfer settlement.

    Lesson: It is not necessarily best to combine smaller groups first.

    In general, all partitions must be tried; there is no shortcut.

References

[1] Tom Verhoeff. "Settling Multiple Debts Efficiently: An Invitation to Computing Science", *Informatics in Education*, **3**(1):105–126 (2004).